



Cornell University



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Reduced Basis Method for the Wave Equation

Crucial Issues for Model Reduction (among others)

Reduced Basis (RB) in a nutshell:

- Want to reduce parametric problem $P(\mu)$
- Get detailed solution $P^N(\mu)$
- Generate reduced problem $P_N(\mu)$
 - Greedy Algorithm
 - Error estimate $\Delta_N(\mu)$
- Compute solution of reduced problem $P_N(\mu)$

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Crucial Issues:

- How good is the RB error estimate, error/residual relation?
- How stable is the RB problem?
- How well is problem reducible?

Reducibility

Kolmogorov N-width:

$$d_N(\mathcal{P}) := \inf_{V_N; \dim(V_N) = N} \sup_{\mu \in \mathcal{P}} \inf_{v_N \in V_N} \|u(\mu) - v_N\|$$

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What do we have to expect?

- Elliptic, coercive: $d_N(\mathcal{P}) \leq e^{-\alpha N}$, $\alpha > 0$ (Maday, Patera, Turinici (2002), Ohlberger, Rave (2016), Buffa, Maday, Patera, Prud'homme, Turinici (2012), Binev, Cohen, Dahmen, DeVore, Petrova, Wojtaszczyk (2011), ...)

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- Linear transport: $d_N(\mathcal{P}) \geq \frac{1}{2}N^{-1/2}$ (Ohlberger, Rave (2016))

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- Linear transport: $d_N(\mathcal{P}) \geq \frac{1}{2}N^{-1/2}$ (Ohlberger, Rave (2016))
- Model reduction/RB for transport problems: (Abgrall, Amsallem, Crisovan (2016), Billaud-Friess, Nouy, Zahm (2014), Cagniart, Crisovan, Maday, Abgrall (2017), Cagniart, Maday, Stamm (2016), Carlberg (2015), Dahmen, Plesken, Welper (2014), Gerbeau, Lombardi (2014), Iollo, Lombardi (2014), Ohlberger, Rave (2013), Reiss et al. (2015), Rim, Peherstorfer, Mandli (2019), Taddei, Perotto, Quarteroni (2013), Welper (2017), Zahm, Nouy (2016), ...)

Here: interested in Wave Equation

Problem definition: find $u(t) \in H$, $t \in I$, such that

$$\begin{aligned}\ddot{u}(t) + D\dot{u}(t) + Au(t) &= g(t) \text{ in } V', \quad t \in I \text{ a.e.,} \\ u(0) &= u_0 \in H, \quad \dot{u}(0) = v_0 \in V'.\end{aligned}$$

Setting:

- $I := [0, T]$, $T > 0$,
- Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$,
- symmetric, bounded, positive operators $A, D \in \mathcal{L}(V, V')$ given by $\langle A\psi, \xi \rangle_{V' \times V} = a(\psi, \xi)$, $\langle D\psi, \xi \rangle_{V' \times V} = d(\psi, \xi)$ for $\psi, \xi \in V$,
- constants α_a, γ_a with $\alpha_a \|\psi\|_V \leq \|A\psi\|_{V'} \leq \gamma_a \|\psi\|_V$, $\psi \in V$,

Outlook:

1. Motivation
2. Second order Wave Equation
3. First order Wave Equation
4. Conclusion

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2nd order Wave Equation

Setting:

- $I := [0, T]$, $T > 0$,
- $A : V \rightarrow V'$, ($A = -\Delta$, $V = H_0^1(\Omega)$)
- $\alpha_a \|\psi\|_V \leq \|A\psi\|_{V'} \leq \gamma_a \|\psi\|_V$, $\psi \in V$

Derive semi-weak 2nd order form:

$$\begin{aligned} \langle \ddot{u}(t), v \rangle_{V' \times V} + a(u(t), v) &= \langle g(t), v \rangle_{V' \times V} \quad \forall v \in V, t \in I \text{ a.e.,} \\ u(0) &= u_0, \quad \dot{u}(0) = u_1. \end{aligned}$$

Well-posedness

- (Wloka, 1987): $g \in L_2(I; H)$, $u_0 \in V$ and $v_0 \in H$
 \rightsquigarrow solution $u \in H^1(I; H) \cap L_2(I; V)$.

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- (Lions, Magenes, 1972): $g \in L_2(I; V')$, $u_0 \in H$ and $v_0 \in V'$,
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A stability Estimate [1]

Lemma (G., Patera, Urban, 2019)

Let $g \in L_2(I; V')$, $u_0 \in H$, $v_0 \in V'$, then

$$\sqrt{\|\dot{u}(t)\|_{V'}^2 + \alpha_a \|u(t)\|_H^2} \leq \sqrt{\|\dot{u}(0)\|_{V'}^2 + \gamma_a \|u(0)\|_H^2} + \int_0^t \|g(s)\|_{V'} ds.$$

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Sketch of proof:

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A detailed 2nd order (semi-)discretization

Discrete spaces of dimension \mathcal{N} :

$$H_h := \text{span}\{h_1, \dots, h_{\mathcal{N}}\} \subset H, \quad V_h := \text{span}\{v_1, \dots, v_{\mathcal{N}}\} \subset V,$$

Detailed approximation $u_h(t)$:

$$\langle \ddot{u}_h(t), v_h \rangle_{V' \times V} + a(u_h(t), v_h) = \langle g(t), v_h \rangle_{V' \times V} \quad \forall v_h \in V_h,$$

$$u_h(0) = u_{0,h}, \quad \dot{u}_h(0) = u_{1,h},$$

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Corollary (Error/residual relation, (G., Patera, Urban, 2019))

For **error** $e_h(t) := u(t) - u_h(t)$, **residual** $r_h(t) := g(t) - \ddot{u}_h(t) - A u_h(t)$, we get

$$\|u(t) - u_h(t)\|_H \leq \sqrt{\frac{\gamma_a}{\alpha_a} \|e_h(0)\|_H^2 + \frac{1}{\alpha_a} \|\dot{e}_h(0)\|_{V'}^2} + \frac{1}{\sqrt{\alpha_a}} \int_0^t \|r_h(s)\|_{V'} ds.$$

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- \rightsquigarrow parametric formulation ($\forall v_h \in V_h$, $t \in I$ a.e.)

$$\langle \ddot{u}(t; \mu), v \rangle_{V' \times V} + a(u(t; \mu), v; \mu) = \langle g(t; \mu), v \rangle_{V' \times V} \quad \forall v \in V,$$
$$u(0; \mu) = u_0, \quad \dot{u}(0; \mu) = u_1.$$

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$$\begin{aligned}\langle \ddot{u}_h(t; \mu), v_h \rangle_{V' \times V} + a(u_h(t; \mu), v_h; \mu) &= \langle g(t; \mu), v_h \rangle_{V' \times V} \quad \forall v_h \in V_h, \\ u_h(0; \mu) &= u_{0,h}, \quad \dot{u}_h(0; \mu) = u_{1,h}.\end{aligned}$$

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- fix $\tau := T/K$, $K > 1$ and $t^k := k\tau$, $k = 1, \dots, K - 1$

$$\begin{aligned}\frac{1}{\tau^2}(u_h^{k+1} - 2u_h^k + u_h^{k-1}, v_h)_{L_2} + a(\theta u_h^{k+1} + (1 - 2\theta)u_h^k + \theta u_h^{k-1}, v_h; \mu) \\ = \theta \langle g(t^{k+1}; \mu), v_h \rangle + (1 - 2\theta) \langle g(t^k; \mu), v_h \rangle + \theta \langle g(t^{k-1}; \mu), v_h \rangle \\ =: \langle g_h^k(\mu), v_h \rangle_{V' \times V}, \quad (\theta - \text{scheme})\end{aligned}$$

Error/Residual Relation

- Rewrite as $\mathcal{L}_I u_h^{k+1} = \mathcal{L}_{E_1} u_h^k + \mathcal{L}_{E_2} u_h^{k-1} + b_h^k$
 - implicit: $\mathcal{L}_I = \mathcal{L}_I(\mu) := I_h + \theta\tau^2 A_h(\mu)$
 - explicit: $\mathcal{L}_{E_1} = \mathcal{L}_{E_1}(\mu) := 2I_h - (1 - 2\theta)\tau^2 A_h(\mu)$
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 - $b_h^k := \tau^2 h_h^k(\mu)$
- $V_N \subset V_h$ of dimension $N \ll \mathcal{N}$, $\mathcal{L}_{I,N} := P_N \circ \mathcal{L}_I$

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- RB residual

$$\mathcal{R}_N^{k+1}(\mu) := \frac{1}{\tau^2} \left(\mathcal{L}_{E_2} u_N^{k-1} + \mathcal{L}_{E_1} u_N^k - \mathcal{L}_I u_N^{k+1} + b_h^k \right)$$

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- Residual/Error relation

$$\mathcal{L}_I e_N^{k+1} = \mathcal{L}_{E_2} e_N^{k-1} + \mathcal{L}_{E_1} e_N^k + \tau^2 \mathcal{R}_N^{k+1}$$

Error Estimate

Arrive at (computable!) error estimate

$$\|u_h^k(\mu) - u_N^k(\mu)\|_H \leq \Delta_N^k(\mu)$$

with

$$\Delta_N^k(\mu) := \sqrt{\frac{\gamma_a}{\alpha_a} \|e_N^0(\mu)\|_H^2 + \frac{1}{\alpha_a} \|\dot{e}_N^0(\mu)\|_{V'}^2} + \frac{\tau}{\sqrt{\alpha_a}} \left(\sum_{i=1}^{k-1} \|\mathcal{R}_N^i(\mu)\|_{V'} \right).$$

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How to construct basis: POD-Greedy (Haasdonk, Ohlberger, 2008)

- ① Choose μ_n with error estimator $\Delta_n^K(\mu)$
- ② Apply POD to chosen trajectory \rightsquigarrow add $u^k(\mu_n)$ to basis

What can we expect? - Wave Equation

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What do we have to expect? (Greif,Urbán,2019)

- Linear wave equation: $\frac{1}{4}N^{-1/2} \leq d_N(\mathcal{P}) \leq \frac{1}{2}(N-1)^{-1/2}$,

$$\ddot{u}(t, x; \mu) - \mu^2 u(t, x; \mu) = 0$$

$$u(0, x; \mu) = u_0(x) := \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

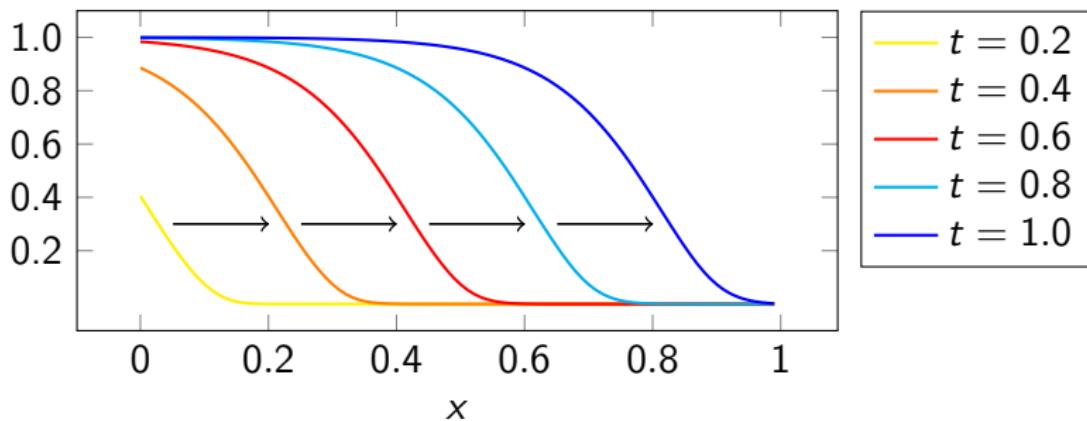
$$\dot{u}(0, x; \mu) = 0$$

$$u(t, -1; \mu) = 1, \quad u(t, 1; \mu) = -1, \quad t \in (0, 1).$$

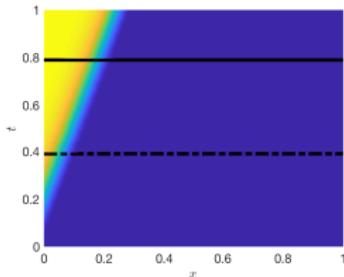
A Numerical Experiment

Setting:

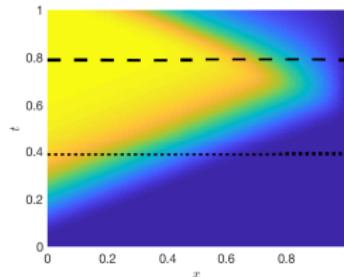
- $\ddot{u}(t, x; \mu) - \mu^2 u_{xx}(t, x; \mu) = 0, \quad t, x \in (0, 1)$
- $u(t, 0; \mu) = \tanh(5t)^3$
- $u(t, 1; \mu) = u_0(x) = u_1(x) = 0$
- $\mu \in \mathcal{P} := [0.3, 2];$
- example $\mu = 1;$



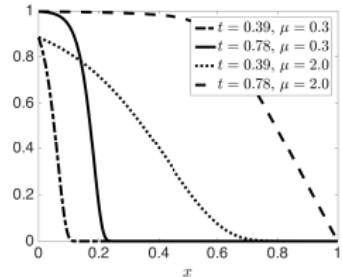
Numerical Example: Detailed Solution



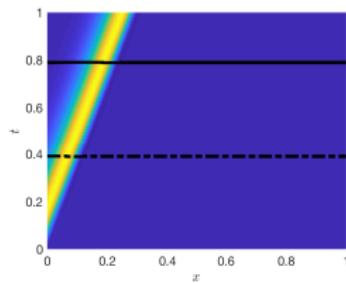
(a) $u(\cdot, \cdot; 0.3)$



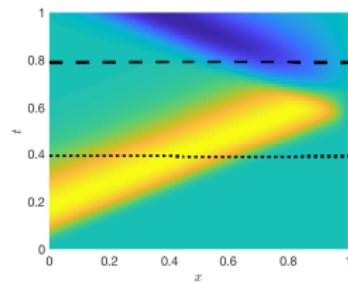
(b) $u(\cdot, \cdot; 2.0)$



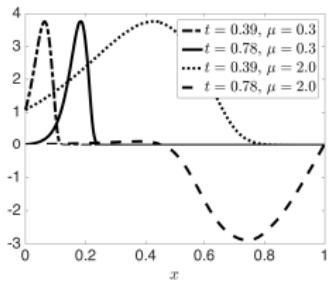
(c) $u(\cdot, \cdot; \mu)$



(d) $u_x(\cdot, \cdot; 0.3)$

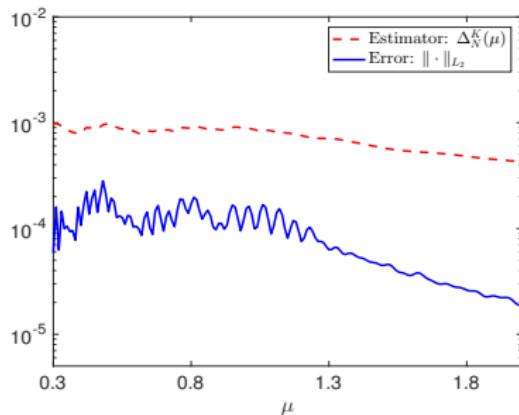
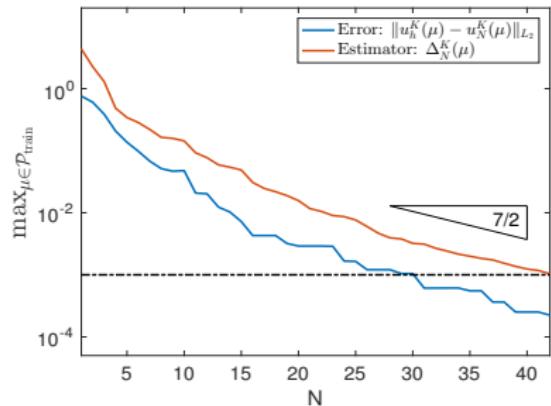


(e) $u_x(\cdot, \cdot; 2.0)$



(f) $u_x(\cdot, \cdot; \mu)$

Greedy Decay and Online Approximation



$$\frac{1}{|\mathcal{P}_{\text{test}}|} \sum_{\mu \in \mathcal{P}_{\text{test}}} \eta_N(\mu) = 10.72, \quad \frac{1}{|\mathcal{P}_{\text{test}} \cap [0.3, 1]|} \sum_{\mu \in \mathcal{P}_{\text{test}} \cap [0.3, 1]} \eta_N(\mu) = 6.92.$$

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2. 2nd order Wave Equation

3. First order Wave Equation

4. Conclusion

First Order System

- 1st order system with damping

$$\begin{aligned}(\dot{u}_1(t), \phi)_H &= (u_2(t), \phi)_H, & \phi \in H, \\ \langle \dot{u}_2(t), \psi \rangle_{V' \times V} &= \langle g(t), \psi \rangle_{V' \times V} - a(u_1(t), \psi) - d(u_2(t), \psi), & \psi \in V.\end{aligned}$$

First Order System

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$$\begin{aligned} (\dot{u}_1(t), \phi)_H &= (f(t), \phi)_H + (u_2(t), \phi)_H, & \phi \in H, \\ \langle \dot{u}_2(t), \psi \rangle_{V' \times V} &= \langle g(t), \psi \rangle_{V' \times V} - a(u_1(t), \psi) - d(u_2(t), \psi), & \psi \in V. \end{aligned}$$

- Include term for deriving error/residual relation

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Lemma (Stability Estimate, (G. Patera, Urban , 2019))

Let $g \in L_2(I; V')$, $f \in L_2(I; H)$, $u_0 \in H$, $v_0 \in V'$, then

$$\begin{aligned} \sqrt{\|u_1(t)\|_H^2 + \frac{1}{\gamma_a} \|u_2(t)\|_{V'}^2} &\leq \sqrt{\|u_1(0)\|_H^2 + \frac{1}{\alpha_a} \|u_2(0)\|_{V'}^2} \\ &+ \max \left\{ 1, \frac{1}{\sqrt{\alpha_a}} \right\} \int_0^t (\|f(s)\|_H^2 + \|g(s)\|_{V'}^2)^{1/2} ds. \end{aligned}$$

A detailed first-order (semi)-discretization

- Discrete spaces of dimension \mathcal{N}
- \rightsquigarrow first order system of dimension $2\mathcal{N}$

Corollary (Error/residual relation, (G., Patera, Urban, 2019))

For *errors* $e_{i,h}(t) := u_i(t) - u_{i,h}(t)$, $i = 1, 2$ and *residuals*

$$r_{1,h}(t) := f(t) - \dot{u}_{1,h} + u_{2,h}(t),$$

$$r_{2,h}(t) := g(t) - \dot{u}_{2,h}(t) - Du_{2,h}(t) - Au_{1,h}(t).$$

we have

$$\begin{aligned} \sqrt{\|e_{1,h}(t)\|_H^2 + \frac{1}{\gamma_a} \|e_{2,h}(t)\|_{V'}^2} &\leq \sqrt{\|e_{1,h}(0)\|_H^2 + \frac{1}{\alpha_a} \|e_{2,h}(0)\|_{V'}^2} \\ &+ \max \left\{ 1, \frac{1}{\sqrt{\alpha_a}} \right\} \int_0^t (\|r_{1,h}(s)\|_H^2 + \|r_{2,h}(s)\|_{V'}^2)^{1/2} ds. \end{aligned}$$

Error/Residual Relation

- Introduce parametric setting with $\mu \in \mathcal{P} \subset \mathbb{R}^P$
- derive **detailed** first order system
- Rewrite as $\mathcal{L}_I u_h^{k+1} = \mathcal{L}_E u_h^k + b_h^k$ with **implicit** and **explicit** part

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- **Residual/Error relation**

$$\mathcal{L}_I e_N^{k+1} = \mathcal{L}_{E_1} e_N^k + \tau \mathcal{R}_N^{k+1}$$

Error Estimate

Arrive at (computable!) error estimate

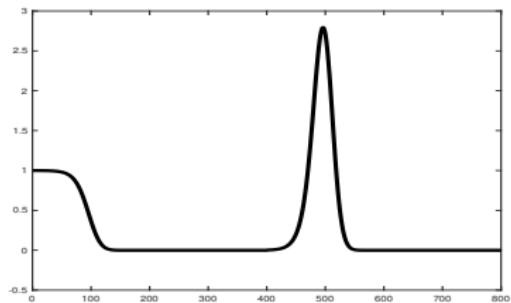
$$\|u_{1,h}(t^k; \mu) - u_{1,N}(t^k; \mu)\|_H \leq \Delta_N^k(\mu), \quad k = 1, \dots, K,$$

with

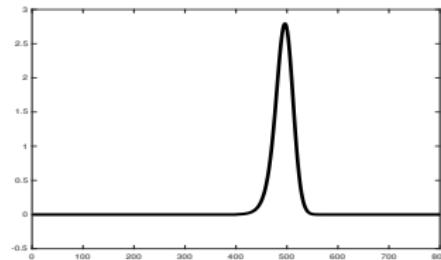
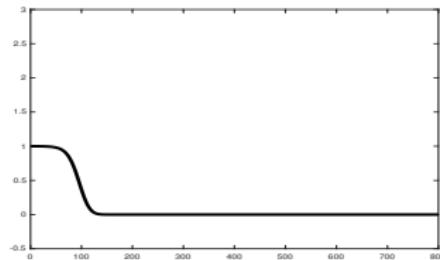
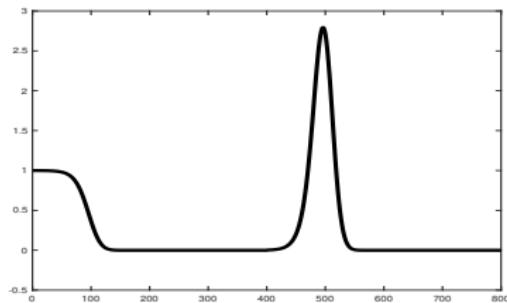
$$\begin{aligned} \Delta_N^k(\mu) := & \sqrt{\|e_{1,N}^0(\mu)\|_H^2 + \frac{1}{\alpha_a} \|e_{2,N}^0(\mu)\|_{V'}^2} \\ & + \tau \max \left\{ 1, \frac{1}{\sqrt{\alpha_a}} \right\} \sum_{i=1}^{k-1} \|\mathcal{R}_N^i(\mu)\|. \end{aligned}$$

- Generate RB with POD-Greedy

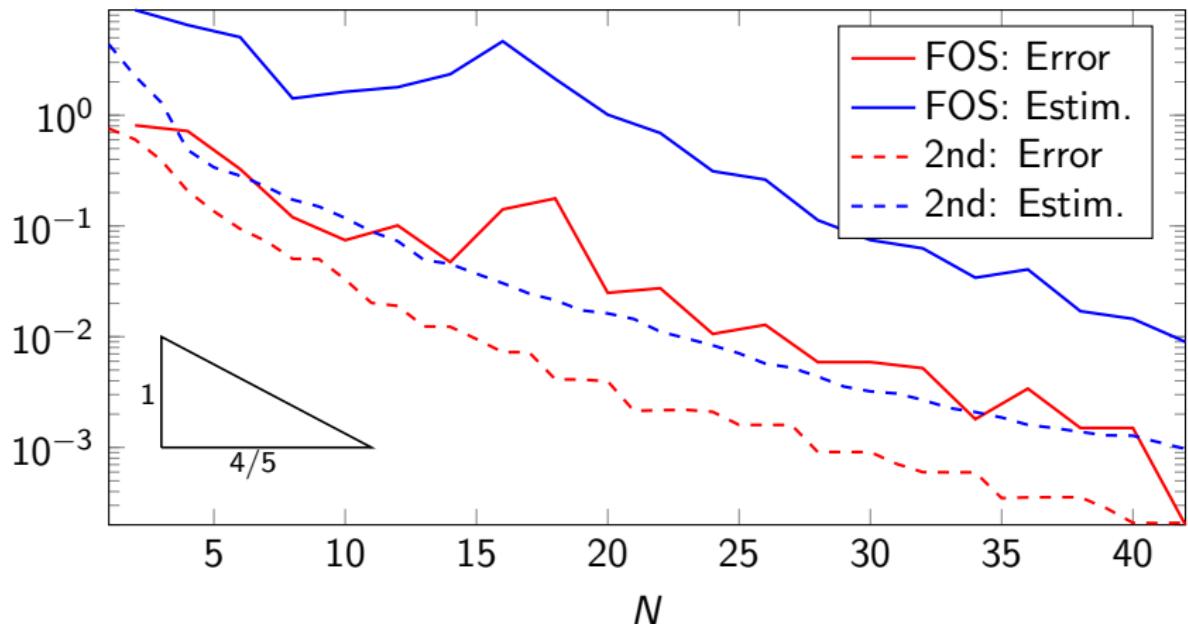
Split Basis $N \rightsquigarrow 2N$



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Greedy Decay and Comparison



Outlook:

1. Motivation
2. 2nd order Wave Equation
3. First order Wave Equation
4. Conclusion

Conclusion

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- Sharp error bound and good effectivities
- Both 2nd order and 1st order system

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- Wave equation space time form (~Brunkens, Smetana, Urban, 2018)
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- Simons Collaboration on Hidden Symmetries and Fusion Energy

